Inverse Trigonometric Functions

Question1

Considering only the principal values of inverse trigonometric functions, the number of positive real values of x satisfying

$\tan^{-1}(x) + \tan^{-1}(2x) = \frac{\pi}{4}$ is :

[27-Jan-2024 Shift 2]

Options:

A.

More than 2

B.

1

C.

2

D.

0

Answer: B

Solution:

$\tan^{-1}x + \tan^{-1}2x = \frac{\pi}{4}; x > 0$

$\Rightarrow \tan^{-1}2x = \frac{\pi}{4} - \tan^{-1}x$

Taking tan both sides

$\Rightarrow 2x = \frac{1-x}{1+x}$

$\Rightarrow 2x^2 + 3x - 1 = 0$

$x = \frac{-3 \pm \sqrt{9+8}}{8} = \frac{-3 \pm \sqrt{17}}{8}$

Only possible $x = \frac{-3+\sqrt{17}}{8}$

Question2

Let $x = m/n$ (m, n are co-prime natural numbers) be a solution of the equation $\cos(2\sin^{-1}x) = 1/9$ and let $\alpha, \beta (\alpha > \beta)$ be the roots of the equation $mx^2 - nx - m + n = 0$. Then the point (α, β) lies on the line

[29-Jan-2024 Shift 2]

Options:

A.

$3x + 2y = 2$

B.

$5x - 8y = -9$

C.

$3x - 2y = -2$

D.

$5x + 8y = 9$

Answer: D

Solution:

Assume $\sin^{-1}x = \theta$

$\cos(2\theta) = \frac{1}{9}$

$\sin\theta = \pm\frac{2}{3}$

as m and n are co-prime natural numbers,

$x = \frac{2}{3}$

i.e. $m = 2, n = 3$

So, the quadratic equation becomes $2x^2 - 3x + 1 = 0$ whose roots are $\alpha = 1, \beta = \frac{1}{2}$

$\left(1, \frac{1}{2}\right)$ lies on $5x + 8y = 9$

Question3

For $\alpha, \beta, \gamma \neq 0$. If $\sin^{-1}\alpha + \sin^{-1}\beta + \sin^{-1}\gamma = \pi$ and $(\alpha+\beta+\gamma)(\alpha-\gamma+\beta) = 3\alpha\beta$, then γ equal to

[31-Jan-2024 Shift 1]

Options:

A.

√3/2

B.

1√2

C.

$\frac{\sqrt{3}-1}{2\sqrt{2}}$

D.

√3

Answer: A

Solution:

Let $sin^{-1}\alpha = A, sin^{-1}\beta = B, sin^{-1}\gamma = C$

$A + B + C = \pi$

$(\alpha + \beta)^2 - \gamma^2 = 3\alpha\beta$

$\alpha^2 + \beta^2 - \gamma^2 = \alpha\beta$

$\frac{\alpha^2 + \beta^2 - \gamma^2}{2\alpha\beta} = \frac{1}{2}$

$\Rightarrow \cos C = \frac{1}{2}$

$sinC = \gamma$

$\cos C = \sqrt{1-\gamma^2} = \frac{1}{2}$

$\gamma = \frac{\sqrt{3}}{2}$

Question4

If $a = \sin^{-1}(\sin(5))$ and $b = \cos^{-1}(\cos(5))$, then $a^2 + b^2$ is equal to

[31-Jan-2024 Shift 2]

Options:

A.

$4\pi^2 + 25$

B.

$8\pi^2 - 40\pi + 50$

C.

$4\pi^2 - 20\pi + 50$

D.

25

Answer: B

Solution:

$$a = \sin^{-1}(\sin 5) = 5 - 2\pi$$

$$\text{and } b = \cos^{-1}(\cos 5) = 2\pi - 5$$

$$\therefore a^2 + b^2 = (5 - 2\pi)^2 + (2\pi - 5)^2$$

$$= 8\pi^2 - 40\pi + 50$$

Question5

$\tan^{-1}\left(\frac{1+\sqrt{3}}{3+\sqrt{3}}\right) + \sec^{-1}\left(\sqrt{\frac{8+4\sqrt{3}}{6+3\sqrt{3}}}\right)$ is equal to

[24-Jan-2023 Shift 1]

Options:

A. $\frac{\pi}{4}$

B. $\frac{\pi}{2}$

C. $\frac{\pi}{3}$

D. $\frac{\pi}{6}$

Answer: C

Solution:

Solution:

$$\tan^{-1}\left(\frac{1+\sqrt{3}}{3+\sqrt{3}}\right) + \sec^{-1}\left(\sqrt{\frac{8+4\sqrt{3}}{6+3\sqrt{3}}}\right)$$

$$= \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) + \sec^{-1}\left(\frac{2}{\sqrt{3}}\right) = \frac{\pi}{3}$$

Question6

If the sum of all the solutions of

$$\tan^{-1}\left(\frac{2x}{1-x^2}\right) + \cot^{-1}\left(\frac{1-x^2}{2x}\right) = \frac{\pi}{3}$$

$-1 < x < 1, x \neq 0$, is $\alpha - \frac{4}{\sqrt{3}}$, then α is equal to ______.

[25-Jan-2023 Shift 1]

Answer: 2

Solution:

Solution:

Case I : $x > 0$

$$\tan^{-1}\frac{2x}{1-x^2} + \tan^{-1}\frac{2x}{1-x^2} = \frac{\pi}{3}$$

$x = 2 - \sqrt{3}$

Case II : $x < 0$

$$\tan^{-1}\frac{2x}{1-x^2} + \tan^{-1}\frac{2x}{1-x^2} + \pi = \frac{\pi}{3}$$

$x = \frac{-1}{\sqrt{3}} \Rightarrow \alpha = 2$

Question7

Let $a_1 = 1, a_2, a_3, a_4, \ldots.$ be consecutive natural numbers. Then

$\tan^{-1}\left(\frac{1}{1+a_1a_2}\right) + \tan^{-1}\left(\frac{1}{1+a_2a_3}\right) + \ldots.. + \tan^{-1}\left(\frac{1}{1+a_{2021}a_{2022}}\right)$ is equal to

[30-Jan-2023 Shift 2]

Options:

A. $\frac{\pi}{4} - \cot^{-1}(2022)$

B. $\cot^{-1}(2022) - \frac{\pi}{4}$

C. $\tan^{-1}(2022) - \frac{\pi}{4}$

D. $\frac{\pi}{4} - \tan^{-1}(2022)$

Answer: 0

Solution:

Sol. $a_2 - a_1 = a_3 - a_2 = \ldots.. = a_{2022} - a_{2021} = 1$.

$\therefore \tan^{-1}\left(\frac{a_2 - a_1}{1 + a_1 a_2}\right) + \tan^{-1}\left(\frac{a_3 - a_2}{1 + a_2 a_3}\right) + \ldots.. + \tan^{-1}\left(\frac{a_{2022} - a_{2021}}{1 + a_{2021} a_{2022}}\right)$

$= [(\tan^{-1} a_2) - \tan^{-1} a_1] + [\tan^{-1} a_3 - \tan^{-1} a_2] + \ldots..$

$+ [\tan^{-1} a_{2022} - \tan^{-1} a_{2021}]$

$= \tan^{-1} a_{2022} - \tan^{-1} a_1$

$= \tan^{-1}(2022) - \tan^{-1} 1 = \tan^{-1} 2022 - \frac{\pi}{4}$ (option 3)

$= \left(\frac{\pi}{2} - \cot^{-1}(2022)\right) - \frac{\pi}{4}$

$= \frac{\pi}{4} - \cot^{-1}(2022)$ (option 1)

Question8

If $\sin^{-1}\frac{\alpha}{17} + \cos^{-1}\frac{4}{5} - \tan^{-1}\frac{77}{36} = 0$, $0 < \alpha < 13$, then $\sin^{-1}(\sin\alpha) + \cos^{-1}(\cos\alpha)$ is equal to
[31-Jan-2023 Shift 1]

Options:

A. π

B. 16

C. 0

D. $16 - 5\pi$

Answer: A

Solution:

Solution:

$\cos^{-1}\frac{4}{5} = \tan^{-1}\frac{3}{4}$

$\therefore \sin^{-1}\frac{\alpha}{17} = \tan^{-1}\frac{77}{36} - \tan^{-1}\frac{3}{4} = \tan^{-1}\left(\frac{\frac{77}{36} - \frac{3}{4}}{1 + \frac{77}{36}\cdot\frac{3}{4}}\right)$

$\sin - 1\frac{\alpha}{17} = \tan^{-1}\frac{8}{15} = \sin^{-1}\frac{8}{17}$

$\Rightarrow \frac{\alpha}{17} = \frac{8}{17} \Rightarrow \alpha = 8$

$\therefore \sin^{-1}(\sin 8) + \cos^{-1}(\cos 8)$

$= 3\pi - 8 + 8 - 2\pi$

$= \pi$

Question9

If the sum and product of four positive consecutive terms of a G.P., are 126 and 1296, respectively, then the sum of common ratios of all such GPs is

[31-Jan-2023 Shift 1]

Options:

A. 7

B. $\frac{9}{2}$

C. 3

D. 14

Answer: A

Solution:

$a, ar, ar^2, ar^3 (a, r > 0)$
$a^4 r^6 = 1296$
$a^2 r^3 = 36$
$a = \frac{6}{r^{3/2}}$
$a + ar + ar^2 + ar^3 = 126$
$\frac{1}{r^{3/2}} + \frac{r}{r^{3/2}} + \frac{r^2}{r^{3/2}} + \frac{r^3}{r^{3/2}} = \frac{126}{6} = 21$
$(r^{-3/2} + r^{3/2}) + (r^{1/2} + r^{-1/2}) = 21$
$r^{1/2} + r^{-1/2} = A$
$r^{-3/2} + r^{3/2} + 3A = A^3$
$A^3 - 3A + A = 21$
$A^3 - 2A = 21$
$A = 3$
$\sqrt{r} + \frac{1}{\sqrt{r}} = 3$
$r_r + 1 = 3\sqrt{r}$
$r^2 + 2r + 1 = 9r$
$r^2 - 7r + 1 = 0$

Question10

Let $(a, b) \subset (0, 2\pi)$ be the largest interval for which $\sin^{-1}(\sin\theta) - \cos^{-1}(\sin\theta) > 0, \theta \in (0, 2\pi)$ holds. If $\alpha x^2 + \beta x + \sin^{-1}(x^2 - 6x + 10) + \cos^{-1}(x^2 - 6x + 10) = 0$
and $\alpha - \beta = b - a$, then α is equal to :
[31-Jan-2023 Shift 2]

Options:

A. $\frac{\pi}{48}$

B. $\frac{\pi}{16}$

C. $\frac{\pi}{8}$

D. $\frac{\pi}{12}$

Answer: D

Solution:

Solution:

$\sin^{-1}\sin\theta - \left(\frac{\pi}{2} - \sin^{-1}\sin\theta\right) > 0$

$\Rightarrow \sin^{-1}\sin\theta > \frac{\pi}{4}$

$\Rightarrow \sin\theta > \frac{1}{\sqrt{2}}$

So, $\theta \in \left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$

$\theta \in \left(\frac{\pi}{4}, \frac{3\pi}{4}\right) = (a, b)$

$\Rightarrow \alpha - a = \frac{\pi}{2} = \alpha - \beta$

$\Rightarrow \beta = \alpha - \frac{\pi}{2}$

$x + \beta x + \sin^{2}[(x-3)^2 + 1] + \cos^{-1}[(x-3)^2 + 1] = 0$

$x = 3,\ 9\alpha + 3\beta + \frac{\pi}{2} + 0 = 0$

$\Rightarrow 9\alpha + 3\left(\alpha - \frac{\pi}{2}\right) + \frac{\pi}{2} = 0$

$\Rightarrow 12\alpha - \pi = 0$

$\alpha = \frac{\pi}{12}$

Question11

Let S be the set of all solutions of the equation $\cos^{-1}(2x) - 2\cos^{-1}\left(\sqrt{1-x^2}\right) = \pi,\ \ x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$. Then $\sum_{x \in S} 2\sin^{-1}(x^2 - 1)$ is equal to

[1-Feb-2023 Shift 1]

Options:

A. 0

B. $\frac{-2\pi}{3}$

C. $\pi - \sin^{-1}\left(\frac{\sqrt{3}}{4}\right)$

D. $\pi - 2\sin^{-1}\left(\frac{\sqrt{3}}{4}\right)$

Answer: B

Solution:

Solution:

$\cos^{-1}(2x) = \pi + 2\cos^{-1}\left(\sqrt{1-x^2}\right)$

LHS $= [0, \pi]$

For equation to be meaningful $\cos^{-1}2x = \pi$ and $\cos^{-1}\left(\sqrt{1-x^2}\right) = 0$ $x = \frac{-1}{2}$ and $x = 0$

which is not possible

$\therefore x \in \varphi$

Now $\Sigma(x) = 0$

$\therefore$ Sum over empty set is always 0

Question12

Let $S = \left\{ x \in R : 0 < x < 1.\text{ and } 2\tan^{-1}\left(\frac{1-x}{1+x}\right) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) \right\}$. If n(S) denotes the number of elements in S then :
[1-Feb-2023 Shift 2]

Options:

A. $n(S) = 2$ and only one element in S is less then $\frac{1}{2}$.

B. $n(S) = 1$ and the element in S is more than $\frac{1}{2}$.

C. $n(S) = 1$ and the element in S is less than $\frac{1}{2}$.

D. $n(S) = 0$

Answer: C

Solution:

Solution:

$0 < x < 1$

$2\tan^{-1}\left(\frac{1-x}{1+x}\right) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

$\tan^{-1}x = \theta \in \left(0, \frac{\pi}{4}\right)$ $\therefore x = \tan\theta$

$2\tan^{-1}\left(\tan\left(\frac{\pi}{4} - \theta\right)\right) = \cos^{-1}(\cos 2\theta)$

$2\left(\frac{\pi}{4} - \theta\right) = 2\theta \therefore 4\theta = \frac{\pi}{2}$ $\therefore \theta = \frac{\pi}{8}$

$x = \tan\frac{\pi}{8} \therefore x = \sqrt{2} - 1 \simeq 0.414$

Question13

If $S = \left\{ x \in R : \sin^{-1}\left(\frac{x+1}{\sqrt{x^2+2x+2}}\right) - \sin^{-1}\left(\frac{x}{\sqrt{x^2+1}}\right) = \frac{\pi}{4} \right\}$, then is equal to ______.
[13-Apr-2023 shift 1]

Answer: 4

Solution:

$$\sin^{-1}\left(\frac{(x+1)}{\sqrt{(x+1)^2+1}}\right)-\sin^{-1}\left(\frac{x}{\sqrt{x^2+1}}\right)=\frac{\pi}{4}$$

$$\because \frac{t}{\sqrt{t^2+1}}\in(-1,1)$$

$$\sin^{-1}\left(\frac{(x+1)}{\sqrt{(x+1)^2+1}}\right)=\sin^{-1}\left(\frac{x}{\sqrt{x^2+1}}\right)+\frac{\pi}{4}$$

$$\frac{(x+1)}{\sqrt{(x+1)^2+1}}=\left(\frac{1}{\sqrt{2}}\right)\cos\left(\sin^{-1}\left(\frac{x}{\sqrt{x^2+1}}\right)\right)+\frac{1}{\sqrt{2}}\left(\frac{x}{\sqrt{x^2+1}}\right)$$

$$=\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{x^2+1}}+\frac{x}{\sqrt{x^2+1}}\right)$$

$$\frac{(x+1)}{\sqrt{(x+1)^2+1}}=\frac{1}{\sqrt{2}}\left(\frac{1+x}{\sqrt{x^2+1}}\right)$$

After solving this equation, we get

$x=-1$ or $x=0$

$S=\{-1,0\}$

$$\sum_{x\in R}\left(\sin\left((x^2+x+5)\frac{\pi}{2}\right)-\cos((x^2+x+5)\pi)\right)$$

$$=\left[\sin\left(\frac{5\pi}{2}\right)-\cos(5\pi)\right]+\left[\sin\left(\frac{5\pi}{2}\right)-\cos(5\pi)\right]=4$$

Question14

For $x\in(-1,1]$, the number of solutions of the equation $\sin^{-1}x=2\tan^{-1}x$ is equal to ________.
[13-Apr-2023 shift 2]

Answer: 2

Solution:

$$\sin^{-1}x=2\tan^{-1}x$$

$$\sin^{-1}x=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow x=\frac{2x}{1+x^2}$$

$$\Rightarrow x\left(\frac{2}{1+x^2}-1\right)=0$$

$\Rightarrow x=0,1,-1$ but -1 is not included.

Answer 2 solutions

Question15

Let $x*y=x^2+y^3$ and $(x*1)*1=x*(1*1)$. Then a value of

$2\sin^{-1}\left(\frac{x^4+x^2-2}{x^4+x^2+2}\right)$ is

[24-Jun-2022-Shift-2]

Options:

A. $\frac{\pi}{4}$

B. $\frac{\pi}{3}$

C. $\frac{\pi}{2}$

D. $\frac{\pi}{6}$

Answer: B

Solution:

Given,

$x * y = x^2 + y^3$

$\therefore x * 1 = x^2 + 1^3 = x^2 + 1$

Now, $(x * 1) * 1 = (x^2 + 1) * 1$

$\Rightarrow (x * 1) * 1 = (x^2 + 1)^2 + 1^3$

$\Rightarrow (x * 1) * 1 = x^4 + 1 + 2x^2 + 1$

Also, $x * (1 * 1)$

$= x * (1^2 + 1^3)$

$= x * 2$

$= x^2 + 2^3$

$= x^2 + 8$

Given that,

$(x * 1) * 1 = x * (1 * 1)$

$\therefore x^4 + 1 + 2x^2 + 1 = x^2 + 8$

$\Rightarrow x^4 + x^2 - 6 = 0$

$\Rightarrow x^4 + 3x^2 - 2x^2 - 6 = 0$

$\Rightarrow x^2(x^2 + 3) - 2(x^3 + 3) = 0$

$\Rightarrow (x^2 + 3)(x^2 - 2) = 0$

$\Rightarrow x^2 = 2, -3$

[$x^2 = -3$. not possible as square of anything should be always possible]

$\therefore x^2 = 2$

$\therefore$ Now,

$$2\sin^{-1}\left(\frac{x^4+x^2-2}{x^4+x^2+2}\right)$$
$$=2\sin^{-1}\left(\frac{2^2+2-2}{2^2+2+2}\right)$$
$$=2\sin^{-1}\left(\frac{4}{8}\right)$$
$$=2\sin^{-1}\left(\frac{1}{2}\right)$$
$$=2\times\frac{\pi}{6}$$
$$=\frac{\pi}{3}$$

Question16

The value of $\tan^{-1}\left(\frac{\cos\left(\frac{15\pi}{4}\right)-1}{\sin\left(\frac{\pi}{4}\right)}\right)$ **is equal to :**

[25-Jun-2022-Shift-2]

Options:

A. $-\frac{\pi}{4}$

B. $-\frac{\pi}{8}$

C. $-\frac{5\pi}{12}$

D. $-\frac{4\pi}{9}$

Answer: B

Solution:

Solution:

$$\tan^{-1}\left(\frac{\cos\left(\frac{15\pi}{4}\right)-1}{\sin\frac{\pi}{4}}\right)$$
$$=\tan^{-1}\left(\frac{\frac{1}{\sqrt{2}}-1}{\frac{1}{\sqrt{2}}}\right)$$
$$=\tan^{-1}(1-\sqrt{2})=-\tan^{-1}(\sqrt{2}-1)$$
$$=-\frac{\pi}{8}$$

Question17

If the inverse trigonometric functions take principal values then $\cos^{-1}\left(\frac{3}{10}\cos\left(\tan^{-1}\left(\frac{4}{3}\right)\right)+\frac{2}{5}\sin\left(\tan^{-1}\left(\frac{4}{3}\right)\right)\right)$ is equal to:
[26-Jun-2022-Shift-2]

Options:

A. 0

B. $\frac{\pi}{4}$

C. $\frac{\pi}{3}$

D. $\frac{\pi}{6}$

Answer: C

Solution:

Solution:

$$\cos^{-1}\left(\frac{3}{10}\cos\left(\tan^{-1}\left(\frac{4}{3}\right)\right)+\frac{2}{5}\sin\left(\tan^{-1}\left(\frac{4}{3}\right)\right)\right)$$

$$=\cos^{-1}\left(\frac{3}{10}\cdot\frac{3}{5}+\frac{2}{5}\cdot\frac{4}{5}\right)$$

$$=\cos^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{3}$$

Question18

$\sin^{-1}\left(\sin\frac{2\pi}{3}\right)+\cos^{-1}\left(\cos\frac{7\pi}{6}\right)+\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$ is equal to:
[27-Jun-2022-Shift-1]

Options:

A. $\frac{11\pi}{12}$

B. $\frac{17\pi}{12}$

C. $\frac{31\pi}{12}$

D. $-\frac{3\pi}{4}$

Answer: A

Solution:

Solution:

$$\sin^{-1}\left(\sin\frac{2\pi}{3}\right)+\cos^{-1}\left(\cos\frac{7\pi}{6}\right)+\tan^{-1}\tan\left(\frac{3\pi}{4}\right)$$

$\sin^{-1}\sin\left(\frac{2\pi}{3}\right) = \pi - \frac{2\pi}{3} = \frac{\pi}{3}$

$\cos^{-1}\left(\cos\frac{2\pi}{6}\right) = 2\pi - \frac{7\pi}{6} = \frac{5\pi}{6}$

$\tan^{-1}\tan\left(\frac{3\pi}{4}\right) = \frac{3\pi}{4} - \pi = \frac{-\pi}{4}$

$\sin^{-1}\left(\sin\frac{2\pi}{3}\right) + \cos^{-1}\cos\frac{7\pi}{6} + \tan^{-1}\tan\frac{3\pi}{4}$

$= \frac{11\pi}{12}$

Question19

The value of $\cot\left(\sum_{n=1}^{50}\tan^{-1}\left(\frac{1}{1+n+n^2}\right)\right)$ is

[27-Jun-2022-Shift-2]

Options:

A. $\frac{26}{25}$

B. $\frac{25}{26}$

C. $\frac{50}{51}$

D. $\frac{52}{51}$

Answer: A

Solution:

Solution:

$\cot\left(\sum_{n=1}^{50}\tan^{-1}\left(\frac{1}{1+n+n^2}\right)\right)$

$= \cot\left(\sum_{n=1}^{50}\tan^{-1}\left(\frac{(n+1)-n}{1+(n+1)n}\right)\right)$

$= \cot\left(\sum_{n=1}^{50}(\tan^{-1}(n+1) - \tan^{-1}n)\right.$.

$= \cot(\tan^{-1}51 - \tan^{-1}1)$

$= \cot\left(\tan^{-1}\left(\frac{51-1}{1+51}\right)\right)$

$= \cot\left(\cot^{-1}\left(\frac{52}{50}\right)\right)$

$= \frac{26}{25}$

Question20

$50\tan\left(3\tan^{-1}\left(\frac{1}{2}\right) + 2\cos^{-1}\left(\frac{1}{\sqrt{5}}\right)\right) + 4\sqrt{2}\tan\left(\frac{1}{2}\tan^{-1}(2\sqrt{2})\right)$ is equal to

[29-Jun-2022-Shift-1]

Answer: 29

Solution:

$$50\tan\left(3\tan^{-1}\frac{1}{2}+2\cos^{-1}\frac{1}{\sqrt{5}}\right)$$
$$+4\sqrt{2}\tan\left(\frac{1}{2}\tan^{-1}2\sqrt{2}\right)$$
$$=50\tan\left(\tan^{-1}\frac{1}{2}+2\left(\tan^{-1}\frac{1}{2}+\tan^{-1}2\right)\right)$$
$$+4\sqrt{2}\tan\left(\frac{1}{2}\tan^{-1}2\sqrt{2}\right)$$
$$=50\tan\left(\tan^{-1}\frac{1}{2}+2\cdot\frac{\pi}{2}\right)\right)+4\sqrt{2}\times\frac{1}{\sqrt{2}}$$
$$=50\left(\tan\tan^{-1}\frac{1}{2}\right)+4$$
$$=25+4=29$$

Question21

The set of all values of k for which $(\tan^{-1}x)^3+(\cot^{-1}x)^3-k\pi^3$, $x\in R_u$ is the interval :
[24-Jun-2022-Shift-1]

Options:

A. $\left[\frac{1}{32},\frac{7}{8}\right)$

B. $\left(\frac{1}{24},\frac{13}{16}\right)$

C. $\left[\frac{1}{48},\frac{13}{16}\right]$

D. $\left[\frac{1}{32},\frac{9}{8}\right)$

Answer: A

Solution:

Solution:

$(\tan^{-1}x)^3 + (\cot^{-1}x)^3 = k\pi^3$

Let $f(t) = t^3 + \left(\frac{\pi}{2} - t\right)^3$

Where $t = \tan^{-1}x;\ x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$= t^3 + \left(\frac{\pi}{2}\right)^3 - \frac{3\pi^2 t}{4} + \frac{3\pi}{2}t^2 - t^3$

$f(t) = \frac{3\pi}{2}t^2 - \frac{3\pi^2}{4} \cdot t + \frac{\pi^3}{8}$

This is a quadratic equation of t.

Here, coefficient of t^2 term is $\frac{3\pi}{2}$ which is >0.

∴ It is a upward parabola.

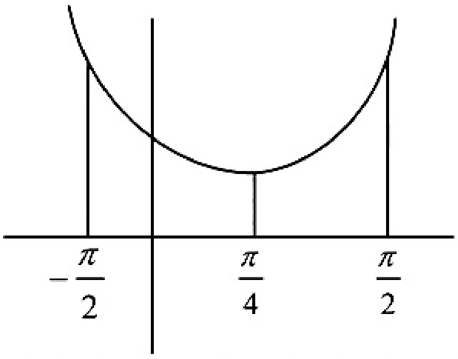
Now, $f'(t) = 3\pi t - \frac{3\pi^2}{4}$

$f''(t) = 3\pi > 0$

$\therefore 3\pi t - \frac{3\pi^2}{4} = 0$

$\Rightarrow t = \frac{\pi}{4}$ (minima)

∴ vertex of graph at $\frac{\pi}{4}$



∴ Minimum value at $\frac{\pi}{4}$ and maximum value at $-\frac{\pi}{2}$.

$\therefore f\left(\frac{\pi}{4}\right) = \frac{\pi^3}{64} + \left(\frac{\pi}{2} - \frac{\pi}{4}\right)^3 = \frac{\pi^3}{32}$

$f\left(-\frac{\pi}{2}\right) = -\frac{\pi^3}{8} + \pi^3$

$= \frac{7\pi^3}{8}$

$\therefore k\pi^3 \in \left[\frac{\pi^3}{32}, \frac{7\pi^3}{8}\right)$

$\Rightarrow k \in \left[\frac{1}{32}, \frac{7}{8}\right)$

Question22

Let $x = \sin(2\tan^{-1}\alpha)$ and $y = \sin\left(\frac{1}{2}\tan^{-1}\frac{4}{3}\right)$. If $S = \{a \in R : y^2 = 1 - x\}$, then $\sum_{\alpha \in S} 16\alpha^3$ is equal to

[25-Jul-2022-Shift-2]

Answer: 130

Solution:

Solution:

$\because x = \sin(2\tan^{-1}\alpha) = \dfrac{2\alpha}{1+\alpha^2}$...... (i)

and $y = \sin\left(\frac{1}{2}\tan^{-1}\frac{4}{3}\right) = \sin\left(\sin^{-1}\frac{1}{\sqrt{5}}\right) = \frac{1}{\sqrt{5}}$

Now, $y^2 = 1 - x$

$\frac{1}{5} = 1 - \dfrac{2\alpha}{1+\alpha^2}$

$\Rightarrow 1 + \alpha^2 = 5 + 5\alpha^2 - 10\alpha$

$\Rightarrow 2\alpha^2 - 5\alpha + 2 = 0$

$\therefore \alpha = 2, \frac{1}{2}$

$\therefore \sum_{\alpha \in S} 16\alpha^3 = 16 \times 2^3 + 16 \times \frac{1}{2^3} = 130$

Question23

$\tan\left(2\tan^{-1}\frac{1}{5} + \sec^{-1}\frac{\sqrt{5}}{2} + 2\tan^{-1}\frac{1}{8}\right)$ is equal to :
[26-Jul-2022-Shift-1]

Options:

A. 1

B. 2

C. $\frac{1}{4}$

D. $\frac{5}{4}$

Answer: B

Solution:

Solution:

$\tan\left(2\tan^{-1}\frac{1}{5} + \sec^{-1}\frac{\sqrt{5}}{2} + 2\tan^{-1}\frac{1}{8}\right)$

$= \tan\left(2\tan^{-1}\left(\dfrac{\frac{1}{5}+\frac{1}{8}}{1-\frac{1}{5}\cdot\frac{1}{8}}\right) + \sec^{-1}\frac{\sqrt{5}}{2}\right)$

$= \tan\left[2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{2}\right]$

$= \tan\left[\tan^{-1}\dfrac{\frac{2}{3}}{1-\frac{1}{9}} + \tan^{-1}\frac{1}{2}\right]$

$= \tan\left[\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{1}{2}\right]$

$= \tan\left[\tan^{-1}\frac{\frac{3}{4}+\frac{1}{2}}{1-\frac{3}{8}}\right] = \tan\left[\tan^{-1}\frac{\frac{5}{4}}{\frac{5}{8}}\right]$

$= \tan[\tan^{-1}2] = 2$

Question24

If $0 < x < \frac{1}{\sqrt{2}}$ and $\frac{\sin^{-1}x}{\alpha} = \frac{\cos^{-1}x}{\beta}$, then the value of $\sin\left(\frac{2\pi\alpha}{\alpha+\beta}\right)$ is [26-Jul-2022-Shift-2]

Options:

A. $4\sqrt{(1-x^2)}(1-2x^2)$

B. $4x\sqrt{(1-x^2)}(1-2x^2)$

C. $2x\sqrt{(1-x^2)}(1-4x^2)$

D. $4\sqrt{(1-x^2)}(1-4x^2)$

Answer: B

Solution:

Solution:

Let $\frac{\sin^{-1}x}{\alpha} = \frac{\cos^{-1}x}{\beta} = k \Rightarrow \sin^{-1}x + \cos^{-1}x = k(\alpha+\beta) \Rightarrow \alpha+\beta = \frac{\pi}{2k}$

Now, $\frac{2\pi\alpha}{\alpha+\beta} = \frac{2\pi\alpha}{\frac{\pi}{2k}} = 4k\alpha = 4\sin^{-1}x$

Here $\sin\left(\frac{2\pi\alpha}{\alpha+\beta}\right) = \sin(4\sin^{-1}x)$

Let $\sin^{-1}x = \theta$

$\because x \in \left(0, \frac{1}{\sqrt{2}}\right) \Rightarrow \theta \in \left(0, \frac{\pi}{4}\right)$

$\Rightarrow x = \sin\theta$

$\Rightarrow \cos\theta = \sqrt{1-x^2}$

$\Rightarrow \sin 2\theta = 2x \cdot \sqrt{1-x^2}$

$\Rightarrow \cos 2\theta = \sqrt{1-4x^2(1-x^2)} = \sqrt{(2x^2-1)^2} = 1-2x^2$

$\because (\cos 2\theta > 0$. as $.2\theta \in \left(0, \frac{\pi}{2}\right))$

$\Rightarrow \sin 4\theta = 2 \cdot 2x\sqrt{1-x^2}(1-2x^2)$

$= 4x\sqrt{1-x^2}(1-2x^2)$

Question25

For $k \in \mathbb{R}$, let the solutions of the equation $\cos(\sin^{-1}(x\cot(\tan^{-1}(\cos(\sin^{-1}x))))) = k$, $0 < |x| < \frac{1}{\sqrt{2}}$ be α and β, where

the inverse trigonometric functions take only principal values. If the solutions of the equation $x^2 - bx - 5 = 0$ are $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$ and $\frac{\alpha}{\beta}$, then $\frac{b}{k^2}$ is equal to ________.
[27-Jul-2022-Shift-1]

Answer: 12

Solution:

Solution:

$\cos(\sin^{-1}(x\cot(\tan^{-1}(\cos(\sin^{-1})))) = k.$

$\Rightarrow \cos\left(\sin^{-1}\left(x\cot\left(\tan^{-1}\sqrt{1-x^2}\right)\right)\right) = k$

$\Rightarrow \cos\left(\sin^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)\right) = k$

$\Rightarrow \frac{\sqrt{1-2x^2}}{\sqrt{1-x^2}} = k$

$\Rightarrow \frac{1-2x^2}{1-x^2} = k^2$

$\Rightarrow 1 - 2x^2 = k^2 - k^2x^2$

$\therefore x^2 - \left(\frac{k^2-1}{k^2-2}\right) = 0 \begin{matrix} / \alpha \\ \backslash \beta \end{matrix}$

$\frac{1}{\alpha^2} + \frac{1}{\beta^2} = 2\left(\frac{k^2-2}{k^2-1}\right) \ldots\ldots (1)$

and $\frac{\alpha}{\beta} = -1 \ldots\ldots (2)$

$\therefore 2\left(\frac{k^2-2}{k^2-1}\right)(-1) = -5$

$\Rightarrow k^2 = \frac{1}{3}$

and $b = S \cdot R = 2\left(\frac{k^2-2}{k^2-1}\right) - 1 = 4$

$\therefore \frac{b}{k^2} = \frac{4}{\frac{1}{3}} = 12$

Question26

Considering the principal values of the inverse trigonometric functions, the sum of all the solutions of the equation $\cos^{-1}(x) - 2\sin^{-1}(x) - \cos^{-1}(2x)$ is equal to :
[28-Jul-2022-Shift-1]

Options:

A. 0

B. 1

C. $\frac{1}{2}$

D. $-\frac{1}{2}$

Answer: A

Solution:

Solution:
$\cos^{-1}x - 2\sin^{-1}x = \cos^{-1}2x$
For Domain : $x \in \left[\frac{-1}{2}, \frac{1}{2}\right]$
$\cos^{-1}x - 2\left(\frac{\pi}{2} - \cos^{-1}x\right) = \cos^{-1}(2x)$
$\Rightarrow \cos^{-1}x + 2\cos^{-1}x = \pi + \cos^{-1}2x$
$\Rightarrow \cos(3\cos^{-1}x) = -\cos(\cos^{-1}2x)$
$\Rightarrow 4x^3 = x$
$\Rightarrow x = 3, \pm\frac{1}{2}$

Question27

f $\frac{\sin^{-1}(x)}{a} = \frac{\cos^{-1}x}{b} = \frac{\tan^{-1}y}{c}$ 0 < x < 1, then the value of $\cos\left(\frac{\pi c}{a+b}\right)$ is
[2021, 26 Feb. Shift-1]

Options:

A. $\frac{1-y^2}{y\sqrt{y}}$

B. $1-y^2$

C. $\frac{1-y^2}{1+y^2}$

D. $\frac{1-y^2}{2y}$

Answer: C

Solution:

Solution:
$\frac{\sin^{-1}x}{a} = \frac{\cos^{-1}x}{b} = \frac{\tan^{-1}y}{c}$... (i)
Take first two terms of Eq. (i)
$\frac{\sin^{-1}x}{a} = \frac{\cos^{-1}x}{b}$
$\Rightarrow \frac{\sin^{-1}x}{a} = \frac{\cos^{-1}x}{b} = \frac{\sin^{-1}x + \cos^{-1}x}{a+b}$
$\Rightarrow \frac{\sin^{-1}x}{a} = \frac{\cos^{-1}x}{b} = \frac{\pi/2}{a+b}$
$\left[\because \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}\right]$
$\Rightarrow \frac{\sin^{-}x}{a} = \frac{\cos^{-1}x}{b} = \frac{\pi/2}{a+b} = \frac{\tan^{-1}y}{c}$
Using last two terms,
$\frac{\tan^{-1}y}{c} = \frac{\pi/2}{a+b}$

$\Rightarrow \tan^{-1}y = \dfrac{\pi c}{2(a+b)}$

$\Rightarrow 2\tan^{-1}y = \dfrac{\pi c}{(a+b)}$

$\Rightarrow \cos^{-1}\left(\dfrac{1-y^2}{1+y^2}\right) = \dfrac{\pi c}{a+b}$

$\left[\because 2\tan^{-1}y = \cos^{-1}\left(\dfrac{1-y^2}{1+y^2}\right)\right]$

$\Rightarrow \dfrac{1-y^2}{1+y^2} = \cos\left(\dfrac{\pi c}{a+b}\right)$

$\therefore \cos\left(\dfrac{\pi c}{a+b}\right) = \dfrac{1-y^2}{1+y^2}$

Question28

$\operatorname{cosec}\left[2\cot^{-1}(5) + \cos^{-1}\left(\frac{4}{5}\right)\right]$ is equal to [2021, 25 Feb. Shift-II]

Options:

A. $\frac{56}{33}$

B. $\frac{65}{33}$

C. $\frac{65}{56}$

D. $\frac{75}{56}$

Answer: C

Solution:

Solution:

$\operatorname{cosec}[2\cot^{-1}(5) + \cos^{-1}(4/5)]$

$= \operatorname{cosec}\left[2\tan^{-1}\left(\frac{1}{5}\right) + \cos^{-1}\left(\frac{4}{5}\right)\right]$

$\left[\because \tan^{-1}x = \cot^{-1}\left(\frac{1}{x}\right)\right]$

$= \operatorname{cosec}\left[\tan^{-1}\left(\dfrac{2\left(\frac{1}{5}\right)}{1-\left(\frac{1}{5}\right)^2}\right) + \cos^{-1}\left(\frac{4}{5}\right)\right]$

$\left[\therefore 2\tan^{-1}\theta = \tan^{-1}\left(\dfrac{2\theta}{1-\theta^2}\right)\right]$

$= \operatorname{cosec}\left(\tan^{-1}\frac{5}{12} + \cos^{-1}\frac{4}{5}\right)$

Let $\tan^{-1}\left(\frac{5}{12}\right) = x$, then $\tan x = \frac{5}{12}$ gives

$\sin x = \frac{5}{13}, \cos x = \frac{12}{13}$

Let $\cos^{-1}\left(\frac{4}{5}\right) = y$, then $\cos y = \frac{4}{5}$ gives,

$\sin y = \frac{3}{5}$

Now, $\operatorname{cosec}(x+y) = \dfrac{1}{\sin(x+y)}$

$$= \frac{1}{\sin x \cos y + \cos x \sin y}$$

$$= \frac{1}{\left(\frac{5}{13}\right)\left(\frac{4}{5}\right) + \left(\frac{12}{13}\right)\left(\frac{3}{5}\right)} = \frac{65}{56}$$

Question29

A possible value of $\tan\left(\frac{1}{4}\sin^{-1}\frac{\sqrt{63}}{8}\right)$ is [2021, 24 Feb. Shift-11]

Options:

A. $\frac{1}{\sqrt{7}}$

B. $2\sqrt{2} - 1$

C. $\sqrt{7} - 1$

D. $\frac{1}{2\sqrt{2}}$

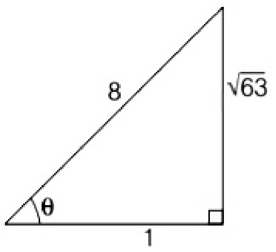
Answer: A

Solution:

Solution:

Given, $\tan\left(\frac{1}{4}\sin^{-1}\frac{\sqrt{63}}{8}\right)$

Let $\sin^{-1}\frac{\sqrt{63}}{8} = \theta$



$\Rightarrow \sin\theta = \frac{\sqrt{63}}{8}$

$\Rightarrow \cos\theta = \frac{1}{8}$

Also, $\cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}}$

$\frac{1+\frac{1}{8}}{2} = \sqrt{\frac{9}{16}} = \frac{3}{4}$

$\therefore \tan\left(\frac{1}{4}\sin^{-1}\frac{\sqrt{63}}{8}\right) = \tan\left(\frac{\theta}{4}\right)$

$$= \sqrt{\frac{1-\cos\frac{\theta}{2}}{1+\cos\frac{\theta}{2}}}$$

$$= \sqrt{\frac{1-\frac{3}{4}}{1+\frac{3}{4}}} = \frac{1}{\sqrt{7}}$$

Question30

If $\cot^{-1}(\alpha) = \cot^{-1}2 + \cot^{-1}8 + \cot^{-1}18 + \cot^{-1}32 + ...$ upto 100 terms, then α is
[2021, 17 March Shift-I]

Options:

A. 1.01

B. 1.00

C. 1.02

D. 1.03

Answer: A

Solution:

Solution:
$\cot^{-1}\alpha = \cot^{-1}2 + \cot^{-1}8 + \cot^{-1}18$
$+\cot^{-1}32 \ldots$ upto 100 terms Let T_n be the nth term of $\cot^{-1}\alpha$.

$T_n = \cot^{-1}(2n^2) = \tan^{-1}\left(\frac{1}{2n^2}\right)$

$= \tan^{-1}\left[\frac{(2n+1)-(2n-1)}{1+(2n+1)(2n-1)}\right]$

$\because \tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$

$\therefore \tan^{-1}\left\{\frac{(2n+1)-(2n-1)}{1+(2n+1)(2n-1)}\right\}$

$= \tan^{-1}(2n+1) - \tan^{-1}(2n-1)$

$T_1 = \tan^{-1}3 - \tan^{-1}1$

$T_2 = \tan^{-1}5 - \tan^{-1}3$

$T_3 = \tan^{-1}7 - \tan^{-1}5$

$\vdots$

$T_{99} = \tan^{-1}199 - \tan^{-1}197$

$T_{100} = \tan^{-1}201 - \tan^{-1}199$

$\sum T_r = \tan^{-1}201 - \tan^{-1}1$

$= \tan^{-1}\left(\frac{201-1}{1+201\cdot 1}\right) = \tan^{-1}\left(\frac{200}{202}\right)$

$\Rightarrow \cot^{-1}(\alpha) = \tan^{-1}\left(\frac{200}{202}\right)$

$\Rightarrow \cot^{-1}(\alpha) = \cot^{-1}\left(\frac{202}{200}\right)$

$\therefore \alpha = \frac{202}{200} = \frac{101}{100} = 1.01$

Question31

The sum of possible values of x for

$\tan^{-1}(x+1) + \cot^{-1}\left(\frac{1}{x-1}\right) = \tan^{-1}\left(\frac{8}{31}\right)$ is

[2021, 17 March Shift-1]

Options:

A. $\frac{-32}{4}$

B. $-\frac{31}{4}$

C. $-\frac{30}{4}$

D. $-\frac{33}{4}$

Answer: A

Solution:

Solution:

$\tan^{-1}(x+1) + \cot^{-1}\left(\frac{1}{x-1}\right) = \tan^{-1}\left(\frac{8}{31}\right)$

$\Rightarrow \tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1}(8/31)$

$\Rightarrow \tan^{-1}\left[\frac{(x+1)+(x-1)}{1-(x+1)(x-1)}\right] = \tan^{-1}\left(\frac{8}{31}\right)$

$\Rightarrow \tan^{-1}\left[\frac{2x}{1-(x^2-1)}\right] = \tan^{-1}\left(\frac{8}{31}\right)$

$\Rightarrow \frac{2x}{2-x^2} = \frac{8}{31}$

$\Rightarrow 62x = 16 - 8x^2$

$\Rightarrow 8x^2 + 62x - 16 = 0$

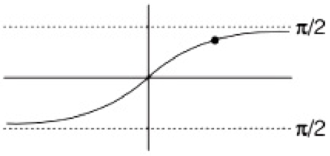
$\Rightarrow 2(x+8)(4x-1) = 0$

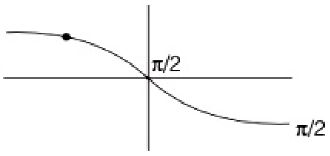
$\Rightarrow x = -8, 1/4$

But at $x = 1/4$

$H\,SS \Rightarrow \tan^{-1}\left(1+\frac{1}{4}\right) + \cot^{-1}\left(\frac{1}{\frac{1}{4}-1}\right)$

$= \tan^{-1}\frac{5}{4} + \cot^{-1}\left(\frac{-4}{3}\right)$





$= \tan^{-1}\frac{5}{4} + \cot^{-1}\left(\frac{-4}{3}\right)_{>\pi/2 > \pi/2}$

$\therefore LH\,S > \pi/2$

$RH\,S = \tan^{-1}\left(\frac{8}{31}\right)_{<\pi/2}$

As, $LH\,S > \pi/2$ and $RH\,S < \pi/2$.

So, $x = -8$ is the only solution.

Question32

The number of solutions of the equation $\sin^{-1}\left[x^2+\frac{1}{3}\right] + \cos^{-1}\left[x^2-\frac{2}{3}\right] = x^2$, for $x \in [-1, 1]$, and $[x]$ denotes the greatest integer less than or equal to x, is

[2021,17 March Shift-II]

Options:

A. 2

B. 0

C. 4

D. infinite

Answer: B

Solution:

Solution:

Given, $\sin^{-1}\left[x^2+\frac{1}{3}\right]+\cos^{-1}\left[x^2-\frac{2}{3}\right]=x^2$

$\sin^{-1}\left[\left(x^2-\frac{2}{3}+1\right)\right]+\cos^{-1}\left[x^2-\frac{2}{3}\right]=x^2$

$\Rightarrow\sin^{-1}\left[\left(x^2-\frac{2}{3}\right)+1\right]+\cos^{-1}\left[x^2-\frac{2}{3}\right]$ . . . (i)

$(\because[x+n]=[x]+n,\ n\in I)$

$\because\left[x^2-\frac{2}{3}\right]$ gives always integral values.

$\therefore\left[x^2-\frac{2}{3}\right]=0,\ -1$ are possible values for $x\in[-1,\ 1]$

$[\because -1\le x\le 1$

$\therefore\ 0\le x^2\le 1$

$\Rightarrow-\frac{2}{3}\le x^2-\frac{2}{3}\le 1-\frac{2}{3}$

$\Rightarrow-\frac{2}{3}\le x^2-\frac{2}{3}\le\frac{1}{3}$

$\Rightarrow-0.66\le x^2-\frac{2}{3}\le 0.33$

$\therefore\ \left[x^2-\frac{2}{3}\right]=-1,\ 0$ are possibilities. $]$

Case I If $\left[x^2-\frac{2}{3}\right]=0$

Then, Eq. (i) becomes,

$\sin^{-1}(1)+\cos^{-1}(0)=x^2$

$\Rightarrow x^2=\pi$

$\Rightarrow x=\pm\sqrt{\pi}$

But at this value of

$x^2,\ \left[x^2-\frac{2}{3}\right]\ne 0$

Case II If $\left[x^2-\frac{2}{3}\right]=-1$

Then, Eq. (i) becomes,

$\sin^{-1}(0)+\cos^{-1}(-1)=x^2$

$\Rightarrow\ x^2=\pi$

$\Rightarrow\ x=\pm\sqrt{\pi}$

But at this value of x^2, $\left[x^2-\frac{2}{3}\right]\ne -1$

$\therefore\ x^2=\pi$

(Rejected also)

Hence, there is no solution for Eq. (i).

$\therefore$ Total number of solution (s) = 0

Question33

Let $S_k = \sum_{r=1}^{k} \tan^{-1}\left(\frac{6^r}{2^{2r+1}+3^{2r+1}}\right)$.
Then, $\lim_{k\to\infty} S_k$ is equal to
[2021, 16 March Shift-1]

Options:

A. $\tan^{-1}\left(\frac{3}{2}\right)$

B. $\frac{\pi}{2}$

C. $\cot^{-1}\left(\frac{3}{2}\right)$

D. $\tan^{-1}(3)$

Answer: C

Solution:

Solution:

$$S_K = \sum_{r=1}^{k} \tan^{-1}\left(\frac{6^r}{2^{2r+1}+3^{2r+1}}\right)$$

$$= \sum \tan^{-1} \frac{2^r 3^{r+1} - 3^r 2^{r+1}}{2^{2r+1}\left[1+\left(\frac{3}{2}\right)^{2r+1}\right]}$$

$$= \sum \tan^{-1} \frac{2^{2r+1}\left[\left(\frac{3}{2}\right)^{r+1} - \left(\frac{3}{2}\right)^{r}\right]}{2^{2r+1}\left[1+\left(\frac{3}{2}\right)^{2r+1}\right]}$$

$$= \sum_{r=1}^{k} \tan^{-1} \frac{\left(\frac{3}{2}\right)^{r+1} - \left(\frac{3}{2}\right)^{r}}{1+\left(\frac{3}{2}\right)^{2r+1}}$$

$$= \sum_{r=1}^{k} \tan^{-1}\left(\frac{3}{2}\right)^{r+1} - \tan^{-1}\left(\frac{3}{2}\right)^{r}$$

$$= \tan^{-1}\left(\frac{3}{2}\right)^{2} - \tan^{-1}\left(\frac{3}{2}\right)$$

$$\tan^{-1}\left(\frac{3}{2}\right)^{3} - \tan^{-1}\left(\frac{3}{2}\right)^{2}$$

$\vdots$

$$\tan^{-1}\left(\frac{3}{2}\right)^{k+1} - \tan^{-1}\left(\frac{3}{2}\right)^{k}$$

$$S_k = \tan^{-1}\left(\frac{3}{2}\right)^{k+1} - \tan^{-1}\left(\frac{3}{2}\right)$$

When $k \to \infty$, $\tan^{-1}\left(\frac{3}{2}\right)^{k+1} \to \pi/2$

$$\lim_{k\to\infty} s_k = \frac{\pi}{2} - \tan^{-1}\left(\frac{3}{2}\right) = \cot^{-1}\left(\frac{3}{2}\right)$$

Question34

Given that the inverse trigonometric functions take principal values only. Then, the number of real values of x which satisfy

$\sin^{-1}\left(\frac{3x}{2}\right) + \sin^{-1}\left(\frac{4x}{5}\right) = \sin^{-1}x$ is equal to
[2021, 16 March Shift-II]

Options:

A. 2

B. 1

C. 3

D. 0

Answer: C

Solution:

Solution:
Given, $\sin^{-1}\left(\frac{3x}{5}\right) + \sin^{-1}\left(\frac{4x}{5}\right) = \sin^{-1}x$
$\Rightarrow \sin^{-1}\left(\frac{3x}{5}\sqrt{1-\frac{16x^2}{25}} + \frac{4x}{5}\sqrt{1-\frac{9x^2}{25}}\right)$
$= \sin^{-1}x$
$\Rightarrow 3x\sqrt{25-16x^2} + 4x\sqrt{25-9x^2} = 25x$
$\Rightarrow \sqrt{225x^2-114x^4} + \sqrt{400x^2-144x^4}$
$= 25x$
$= 25x$ (i)
$(400x^2-144x^4)-(225x^2-144x^4) = 175x^2...$ (ii)
On dividing Eq. (ii) by Eq. (i),
$\sqrt{400x^2-144x^4} - \sqrt{225x^2-144x^2}$
$= 7x...$ (iii)
Now, adding Eqs. (i) and (iii),
$2\sqrt{400x^2-144x^4} = 32x$
$\Rightarrow 400x^2-144x^4 = 256x^2$
$\Rightarrow 144x^2-144x^4 = 0$
$\Rightarrow 144x^2(1-x^2) = 0$
$x = 0, -1, 1$
Hence, 3 real values for x satisfies the equation.

Question35

The number of real roots of the equation $\tan^{-1}\sqrt{x(x+1)} + \sin^{-1}\sqrt{x^2+x+1} = \frac{\pi}{4}$ is
[2021, 20 July Shift-1]

Options:

A. 1

B. 4

C. 3

D. 0

Answer: D

Solution:

$\tan^{-1}\sqrt{x(x+1)} + \sin^{-1}\sqrt{x^2+x+1} = \frac{\pi}{4}$

Domain, $x(x+1) \geq 0$

$0 \leq x^2 + x + 1 \leq 1$

So, only when $x^2 + x = 0$, equation will be define(d)

$x = 0, -1$

At $x = 0$, $\tan^{-1}0 + \sin^{-1}1 = \frac{\pi}{2}$

$x = -1$, $\tan^{-1}0 + \sin^{-1}1 = \frac{\pi}{2}$

∴ No solution.

Question36

The value of $\tan\left(2\tan^{-1}\left(\frac{3}{5}\right) + \sin^{-1}\left(\frac{5}{13}\right)\right)$ is equal to [2021, 20 July Shift-II]

Options:

A. $\frac{-181}{69}$

B. $\frac{220}{21}$

C. $\frac{-291}{76}$

D. $\frac{151}{63}$

Answer: B

Solution:

Solution:

Let

$A = \tan\left(2\tan^{-1}\left(\frac{3}{5}\right) + \sin^{-1}\left(\frac{5}{13}\right)\right)$ . . . (i)

Now, using $2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}$

$2\tan^{-1}\left(\frac{3}{5}\right) = \tan^{-1}\left(\frac{2\left(\frac{3}{5}\right)}{1-\left(\frac{3}{5}\right)^2}\right)$

$= \tan^{-1}\left(\frac{\frac{6}{5}}{1-\frac{9}{25}}\right)$

$= \tan^{-1}\left(\frac{30}{16}\right) = \tan^{-1}\left(\frac{15}{8}\right)$

Let $\sin^{-1}\left(\frac{5}{13}\right) = \theta$, then $\sin\theta = \frac{5}{13}$

$\therefore \tan\theta = \frac{\sin\theta}{\sqrt{1-\sin^2\theta}} = \frac{\frac{5}{13}}{\sqrt{1-\left(\frac{5}{3}\right)^2}} = \frac{\frac{5}{13}}{\frac{12}{13}}$

$\Rightarrow\ \theta = \tan^{-1}\left(\frac{5}{12}\right) = \sin^{-1}\left(\frac{5}{13}\right)$

From Eq. (i),

$A = \tan\left(\tan^{-1}\left(\frac{15}{8}\right) + \tan^{-1}\left(\frac{5}{12}\right)\right)$

$= \tan\left[\tan^{-1}\left(\frac{\frac{15}{8}+\frac{5}{12}}{1-\left(\frac{15}{8}\right)\left(\frac{5}{12}\right)}\right)\right]$

$= \tan\left[\tan^{-1}\left(\frac{\frac{55}{24}}{\frac{21}{96}}\right)\right]$

$= \tan\left[\tan^{-1}\frac{55\times 4}{21}\right]$

$= \tan\left(\tan^{-1}\frac{220}{21}\right) = \frac{220}{21}$

Question37

If $(\sin^{-1}x)^2 - (\cos^{-1}x)^2 = a$, $0 < x < 1$, $a \neq 0$, then the value of $2x^2 - 1$ is [2021, 27 Aug. Shift-1]

Options:

A. $\cos\left(\frac{4a}{\pi}\right)$

B. $\sin\left(\frac{2a}{\pi}\right)$

C. $\cos\left(\frac{2a}{\pi}\right)$

D. $\sin\left(\frac{4a}{\pi}\right)$

Answer: B

Solution:

Solution:

Given, $(\sin^{-1}x)^2 - (\cos^{-1}x)^2 = a$

$\Rightarrow\ (\sin^{-1}x + \cos^{-1}x)(\sin^{-1}x - \cos^{-1}x) = a$

$\Rightarrow\ \frac{\pi}{2}(\sin^{-1}x - \cos^{-1}x) = a$

$\Rightarrow\ \frac{\pi}{2} - 2\cos^{-1}x = \frac{2a}{\pi}$

$\Rightarrow\ 2\cos^{-1}x = \frac{\pi}{2} - \frac{2a}{\pi}$

$\Rightarrow\ \cos^{-1}(2x^2 - 1) = \frac{\pi}{2} - \frac{2a}{\pi}$

$\Rightarrow\ 2x^2 - 1 = \cos\left(\frac{\pi}{2} - \frac{2a}{\pi}\right)$

$\Rightarrow\ 2x^2 - 1 = \sin\left(\frac{2a}{\pi}\right)$

Question38

Let M and m respectively be the maximum and minimum values of the function $f(x) = \tan^{-1}(\sin x + \cos x)$ in $\left[0, \frac{\pi}{2}\right]$, then the value of $\tan(M - m)$ is
[2021,27 Aug. Shift-II]

Options:

A. $2 + \sqrt{3}$

B. $2 - \sqrt{3}$

C. $3 + 2\sqrt{2}$

D. $3 - 2\sqrt{2}$

Answer: D

Solution:

Solution:
We have, $f(x) = \tan^{-1}(\sin x + \cos x)$
$\because \ x \in \left[0, \frac{\pi}{2}\right]$
$\Rightarrow 1 \le \sin x + \cos x \le \sqrt{2}$
$\left[\because -\sqrt{A^2 + B^2} \le \le A\sin x + B\cos x \le \sqrt{A^2 + B^2}\right]$
$\Rightarrow \tan^{-1}(1) \le \tan^{-1}(\sin x + \cos x) \le \tan^{-1}(\sqrt{2})$
$\therefore \ m = \tan^{-1}(1)$ and $M = \tan^{-1}(\sqrt{2})$
$\therefore \ M - m = \tan^{-1}\sqrt{2} - \tan^{-1}(\sqrt{1})$
$= \tan^{-1}\left(\frac{\sqrt{2} - 1}{1 + \sqrt{2}}\right)$
$= \tan^{-1}(3 - 2\sqrt{2})$

Question39

If $y(x) = \cot^{-1}\left(\frac{\sqrt{1 + \sin x} + \sqrt{1 - \sin x}}{\sqrt{1 + \sin x} - \sqrt{1 - \sin x}}\right)$ $\cdot x \in \left(\frac{\pi}{2}, \pi\right)$, then $\frac{dy}{dx}$ at $x = \frac{5\pi}{6}$ is
[2021, 27 Aug. Shift-II]

Options:

A. $-\frac{1}{2}$

B. -1

C. $\frac{1}{2}$

D. 0

Answer: A

Solution:

Solution:

$y(x) = \cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)$

$x \in \left(\frac{\pi}{2}, \pi\right)$

$= \cot^{-1}\frac{\left|\cos\frac{x}{2}+\sin\frac{x}{2}\right|+\left|\cos\frac{x}{2}-\sin\frac{x}{2}\right|}{\left|\cos\frac{x}{2}+\sin\frac{x}{2}\right|-\left|\cos\frac{x}{2}-\sin\frac{x}{2}\right|}$

$= \cot^{-1}\left(\frac{\cos\frac{x}{2}+\sin\frac{x}{2}+\sin\frac{x}{2}-\cos\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}-\sin\frac{x}{2}+\cos\frac{x}{2}}\right)$

$\left[\because \frac{x}{2} \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)\right]$

$= \cot^{-1}\left(\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\right) = \cot^{-1}\left(\tan\frac{x}{2}\right)$

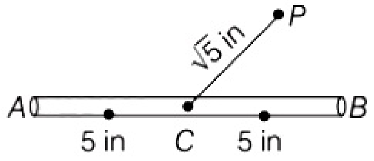
$= \cot^{-1}\left(\cot\left(\frac{\pi}{2}-\frac{x}{2}\right)\right)$

$\Rightarrow y(x) = \frac{\pi}{2}-\frac{x}{2}$

$\therefore y'(x) = \frac{-1}{2}$

Question40

A 10 inches long pencil AB with mid-point C and a small eraser P are placed on the horizontal top of a table such that PC = $\sqrt{5}$ inches and $\angle PCB = \tan^{-1}(2)$. The acute angle through which the pencil must be rotated about C so that the perpendicular distance between eraser and pencil becomes exactly 1 inch is



[2021, 26 Aug. Shift-II]

Options:

A. $\tan^{-1}\left(\frac{3}{4}\right)$

B. $\tan^{-1}(1)$

C. $\tan^{-1}\left(\frac{4}{3}\right)$

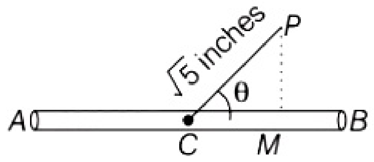
D. $\tan^{-1}\left(\frac{1}{2}\right)$

Answer: A

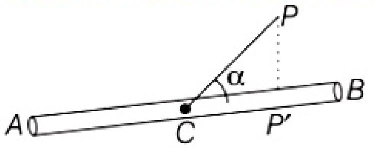
Solution:

Solution:
Let the angle before rotation be θ , then

$\theta = \angle PCB = \tan^{-1}(2)$



Now, after rotation let angle become α .



In $\triangle PCM$
$PC = \sqrt{5}$
$PM = \sqrt{5}\sin\theta = \sqrt{5}\left(\frac{2}{\sqrt{5}}\right) = 2$
After rotation perpendicular distance becomes $PP' = 1$
$PC\sin\alpha = 1$
$\sqrt{5}\sin\alpha = 1$
$\Rightarrow \sqrt{5}\sin\alpha = 1$
$\Rightarrow \sin\alpha = \frac{1}{\sqrt{5}}$
$\Rightarrow \tan\alpha = \frac{\sin\alpha}{\sqrt{1-\sin^2\alpha}} = \frac{1}{2}$
∴ Rotated angle
$= \theta - \alpha = \tan^{-1}(2) - \tan^{-1}\left(\frac{1}{2}\right)$
$= \tan^{-1}\left(\frac{2-1/2}{1+2\times1/2}\right) = \tan^{-1}\left(\frac{3}{4}\right)$

Question41

$\cos^{-1}(\cos(-5)) + \sin^{-1}(\sin(6)) - \tan^{-1}(\tan(12))$ is equal to (The inverse trigonometric functions take the principal values)
[2021, 01 Sep. Shift-2]

Options:

A. $3\pi - 11$

B. $4\pi - 9$

C. $4\pi - 11$

D. $3\pi + 1$

Answer: C

Solution:

Solution:
$\cos^{-1}(\cos(-5)) + \sin^{-1}(\sin(6))$
$-\tan^{-1}(\tan(12))$
$= 2\pi - 5 + (-2\pi + 6) - (12 - 4\pi)$
$= 4\pi - 11$

Question42

$2\pi - \left(\sin^{-1}\frac{4}{5} + \sin^{-1}\frac{5}{13} + \sin^{-1}\frac{16}{65}\right)$ is equal to:
[Sep. 03, 2020 (I)]

Options:

A. $\frac{\pi}{2}$

B. $\frac{5\pi}{4}$

C. $\frac{3\pi}{2}$

D. $\frac{7\pi}{4}$

Answer: C

Solution:

Solution:

$2\pi - \left(\sin^{-1}\frac{4}{5} + \sin^{-1}\frac{5}{13} + \sin^{-1}\frac{16}{65}\right)$

$= 2\pi - \left(\tan^{-1}\frac{4}{3} + \tan^{-1}\frac{5}{12} + \tan^{-1}\frac{16}{63}\right)$ $\left[\because \sin^{-1}\frac{4}{5} = \tan^{-1}\frac{4}{3}\right]$

$= 2\pi - \left\{ \tan^{-1}\left(\frac{\frac{4}{3} + \frac{5}{12}}{1 - \frac{4}{3}\cdot\frac{5}{12}}\right) + \tan^{-1}\frac{16}{63} \right\}$

$= 2\pi - \left(\tan^{-1}\frac{63}{16} + \tan^{-1}\frac{16}{63}\right)$

$= 2\pi - \left(\tan^{-1}\frac{63}{16} + \cot^{-1}\frac{63}{16}\right)$

$= 2\pi - \frac{\pi}{2} = \frac{3\pi}{2}.$

Question43

If S is the sum of the first 10 terms of the seriestan$^{-1}\left(\frac{1}{3}\right)$ + tan$^{-1}\left(\frac{1}{7}\right)$ + tan$^{-1}\left(\frac{1}{13}\right)$ +tan$^{-1}\left(\frac{1}{21}\right)$ + ...then tan(S) is equal to:
[Sep. 05, 2020 (I)]

Options:

A. $\frac{5}{6}$

B. $\frac{5}{11}$

C. $-\frac{6}{5}$

D. $\frac{10}{11}$

Answer: A

Solution:

Solution:

$S = \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{13} + \ldots\ldots$ upto 10 terms

$= \tan^{-1}(2 - 11 + 2\,.\,1) + \tan^{-1}\left(\frac{3-2}{1+3\,.\,2}\right)$

$+\tan^{-1}\left(\frac{4-3}{1+3\,.\,4}\right) + \ldots.. + \tan^{-1}\left(\frac{11-10}{1+11\,.\,10}\right) = (\tan^{-1}2 - \tan^{-1}1) + (\tan^{-1}3 - \tan^{-1}2) +$

$(\tan^{-1}4 - \tan^{-1}3) + \ldots.. + (\tan^{-1}11 - \tan^{-1}10)$

$= \tan^{-1}11 - \tan^{-1}1 \;= \tan^{-1}\left(\frac{11-1}{1+11\,.\,1}\right) = \tan^{-1}\left(\frac{5}{6}\right)$

$\therefore \tan(S) = \frac{5}{6}$

Question44

Considering only the principal values of inverse functions,the set
$A = \left\{ x \geq 0 : \tan^{-1}(2x) + \tan^{-1}(3x) = \frac{\pi}{4} \right\}$
[Jan. 12, 2019 (I)]

Options:

A. contains two elements

B. contains more than two elements

C. is a singleton

D. is an empty set

Answer: C

Solution:

Solution:

Consider, $\tan^{-1}(2x) + \tan^{-1}(3x) = \frac{\pi}{4}$

$\Rightarrow \tan^{-1}\left(\frac{5x}{1-6x^2}\right) = \frac{\pi}{4}$

$\Rightarrow \frac{5x}{1-6x^2} = 1 \Rightarrow 5x = 1 - 6x^2$

$\Rightarrow 6x^2 + 5x - 1 = 0$

$\Rightarrow (6x-1)(x+1) = 0$

$\Rightarrow x = \frac{1}{6}(\text{as } x \geq 0)$

Therefore, A is a singleton set.

Question45

All x satisfying the inequality $(\cot^{-1}x)^2 - 7(\cot^{-1}x) + 10 > 0$, lie in the interval :
[Jan. 11, 2019 (II)]

Options:

A. $(-\infty, \cot 5) \cup (\cot 4, \cot 2)$

B. $(\cot 2, \infty)$

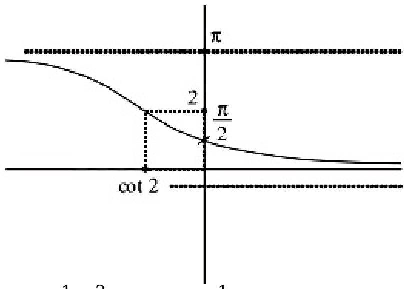
C. $(-\infty, \cot 5) \cup (\cot 2, \infty)$

D. $(\cot 5, \cot 4)$

Answer: B

Solution:

Solution:



$(\cot^{-1}x)^2 - 7(\cot^{-1}x) + 10 > 0$
$(\cot^{-1}x - 5)(\cot^{-1} - 2) > 0$
$\cot^{-1}x \in (-\infty, 2) \cup (5, \infty)$ .....(1)
But $\cot^{-1}x$ lies in $(0, \pi)$
Now, from equation (1)
$\cot^{-1}x \in (0, 2)$
Now, it is clear from the graph $x \in (\cot 2, \infty)$

Question46

The value of $\cot\left(\sum_{n=1}^{19} \cot^{-1}\left(1 + \sum_{p=1}^{n} 2p\right)\right)$ is:

[Jan. 10, 2019 (II)]

Options:

A. $\frac{21}{19}$

B. $\frac{19}{21}$

C. $\frac{22}{23}$

D. $\frac{23}{22}$

Answer: A

Solution:

Solution:

$\cot\left(\sum_{n=1}^{19} \cot^{-1}\left(1 + \sum_{p=1}^{n} 2p\right)\right)$

$= \cot\left(\sum_{n=1}^{19} \cot^{-1}(1 + n(n + 1))\right)$

$= \cot\left(\sum_{n=1}^{19} \tan^{-1}\left(\frac{(n + 1) - n}{1 + (n + 1)n}\right)\right)$ $\left[\cot^{-1}x = \tan^{-1}\left(\frac{1}{x}\right) : \text{for } x > 0\right]$

$= \cot\left(\sum_{n=1}^{19} (\tan^{-1}(n + 1) - \tan^{-1}n)\right)$

$= \cot(\tan^{-1}20 - \tan^{-1}1)$

$= \cot\left(\tan^{-1}\left(\frac{20 - 1}{1 + 20 \times 1}\right)\right)$

$= \cot\left(\tan^{-1}\left(\frac{19}{21}\right)\right) = \cot\cot^{-1}\left(\frac{21}{19}\right) = \frac{21}{19}$

Question47

If $x = \sin^{-1}(\sin 10)$ and $y = \cos^{-1}(\cos 10)$, then $y - x$ is equal to: [Jan. 09, 2019 (II)]

Options:

A. 0

B. 10

C. 7π

D. π

Answer: D

Solution:

Solution:

$x = \sin^{-1}(\sin 10)$

$\Rightarrow x = 3\pi - 10$ $\left\{ \begin{array}{l} 3\pi - \frac{\pi}{2} < 10 < 3\pi + \frac{\pi}{2} \\ \Rightarrow 3\pi - x = 10 \end{array} \right.$

and $y = \cos^{-1}(\cos 10)$ $\left\{ \begin{array}{l} 3\pi < 10 < 4\pi \\ \Rightarrow 4\pi - x = 10 \end{array} \right.$

$\Rightarrow y = 4\pi - 10$

$\therefore y - x = (4\pi - 10) - (3\pi - 10) = \pi$

Question48

If $\cos^{-1}\left(\frac{2}{3x}\right) + \cos^{-1}\left(\frac{3}{4x}\right) = \frac{\pi}{2}\left(x > \frac{3}{4}\right)$, then x is equal to: [Jan. 09, 2019 (I)]

Options:

A. $\frac{\sqrt{145}}{12}$

B. $\frac{\sqrt{145}}{10}$

C. $\frac{\sqrt{146}}{12}$

D. $\frac{\sqrt{145}}{11}$

Answer: A

Solution:

Solution:

$\cos^{-1}\left(\frac{2}{3x}\right) + \cos^{-1}\left(\frac{3}{4x}\right) = \frac{\pi}{2}; \left(x > \frac{3}{4}\right)$

$\Rightarrow \cos^{-1}\left(\frac{2}{3x}\right) = \frac{\pi}{2} - \cos^{-1}\left(\frac{3}{4x}\right)$

$\Rightarrow \cos^{-1}\left(\frac{2}{3x}\right) = \sin^{-1}\left(\frac{3}{4x}\right) \left[\because \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}\right]$

$\sin^{-1}\left(\frac{3}{4x}\right) = \theta \Rightarrow \sin\theta = \frac{3}{4x}$

$\Rightarrow \cos\theta = \sqrt{1 - \sin^2\theta} = \sqrt{1 - \frac{9}{16x^2}}$

$\Rightarrow \theta = \cos^{-1}(\sqrt{16}x^2 - 94x)$

$\therefore \cos^{-1}\left(\frac{2}{3x}\right) = \cos^{-1}\left(\frac{\sqrt{16x^2 - 9}}{4x}\right)$

$\Rightarrow \frac{2}{3x} = \frac{\sqrt{16x^2 - 9}}{4x} \Rightarrow x^2 = \frac{64 + 81}{9 \times 16} \Rightarrow x = \pm\sqrt{\frac{145}{144}}$

$\Rightarrow x = \frac{\sqrt{145}}{12} \quad \left(\because x > \frac{3}{4}\right)$

Question49

If $\alpha = \cos^{-1}\left(\frac{3}{5}\right)$, $\beta = \tan^{-1}\left(\frac{1}{3}\right)$, where $0 < \alpha, \beta < \frac{\pi}{2}$, then $\alpha - \beta$ is equal to:
[April 8, 2019 (I)]

Options:

A. $\tan^{-1}\left(\frac{9}{5\sqrt{10}}\right)$

B. $\cos^{-1}\left(\frac{9}{5\sqrt{10}}\right)$

C. $\tan^{-1}\left(\frac{9}{14}\right)$

D. $\sin^{-1}\left(\frac{9}{5\sqrt{10}}\right)$

Answer: D

Solution:

Solution:

$\because \cos\alpha = \frac{3}{5}$, then $\sin\alpha = \frac{4}{5}$

$\Rightarrow \tan\alpha = \frac{4}{3}$

and $\tan\beta = \frac{1}{3}$

$\because \tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha \,.\, \tan\beta}$

$= \frac{\frac{4}{3} - \frac{1}{3}}{1 + \frac{4}{9}} = \frac{1}{\frac{13}{9}} = \frac{9}{13}$

$\therefore \alpha - \beta = \tan^{-1}\left(\frac{9}{13}\right) = \sin^{-1}\left(\frac{9}{5\sqrt{10}}\right)$

$= \cos^{-1}\left(\frac{13}{5\sqrt{10}}\right)$

Question50

The value of $\sin^{-1}\left(\frac{12}{13}\right) - \sin^{-1}\left(\frac{3}{5}\right)$ is equal to

[April 12, 2019 (I)]

Options:

A. $\pi - \sin^{-1}\left(\frac{63}{65}\right)$

B. $\frac{\pi}{2} - \sin^{-1}\left(\frac{56}{65}\right)$

C. $\frac{\pi}{2} - \cos^{-1}\left(\frac{9}{65}\right)$

D. $\pi - \cos^{-1}\left(\frac{33}{65}\right)$

Answer: B

Solution:

Solution:

$-\sin^{-1}\left(\frac{3}{5}\right) + \sin^{-1}\left(\frac{12}{13}\right) = -\sin^{-1}\left(\frac{3}{5} \times \frac{5}{13} - \frac{12}{13} \times \frac{4}{5}\right)$

($\because$xye"0 and $x^2 + y^2$d "1)

[$\because \sin^{-1}x - \sin^{-1}y = \sin^{-1}\{x\sqrt{1 - y^2} - y\sqrt{1 - x^2}\}$]

$= \sin^{-1}\left(\frac{-33}{65}\right) = \sin^{-1}\left(\frac{33}{65}\right)$

$= \cos^{-1}\left(\frac{56}{65}\right) = \frac{\pi}{2} - \sin^{-1}\left(\frac{56}{65}\right)$

Question51

If $\cos^{-1}x - \cos^{-1}\frac{y}{2} = \alpha$, where $-1 \le x \le 1$, $-2 \le y \le 2$, $x \le \frac{y}{2}$, then for all x, y, $4x^2 - 4xy\cos\alpha + y^2$ is equal to:

[April 10, 2019 (II)]

Options:

A. $4\sin^2\alpha$

B. $2\sin^2\alpha$

C. $4\sin^2\alpha - 2x^2y^2$

D. $4\cos^2\alpha + 2x^2y^2$

Answer: A

Solution:

Solution:

Given, $\cos^{-1}x - \cos^{-1}\frac{y}{2} = \alpha$

$\Rightarrow \cos^{-1}\left(\frac{xy}{2} + \sqrt{1-x^2}\,.\ \sqrt{1-\frac{y^2}{4}}\right) = \alpha$

$\Rightarrow \frac{xy}{2} + \frac{\sqrt{1-x^2}\sqrt{4-y^2}}{2} = \cos\theta$

$\Rightarrow xy + \sqrt{1-x^2}\sqrt{4-y^2} = 2\cos\alpha$

$\Rightarrow (xy - 2\cos\alpha)^2 = (1-x^2)(4-y^2)$

$\Rightarrow x^2y^2 + 4\cos^2\alpha - 4xy\cos\alpha = 4 - y^2 - 4x^2 + x^2y^2$

$\Rightarrow 4x^2 - 4xy\cos\alpha + y^2 = 4\sin^2\alpha$

Question52

A value of x satisfying the equation $\sin[\cot^{-1}(1+x)] = \cos[\tan^{-1}x]$, is : [Online April 9, 2017]

Options:

A. $-\frac{1}{2}$

B. -1

C. 0

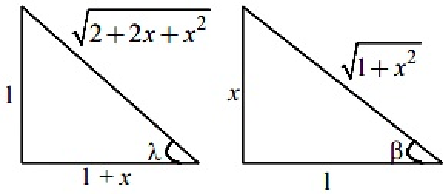
D. $\frac{1}{2}$

Answer: A

Solution:

Solution:

$\sin[\cot^{-1}(1+x)] = \cos(\tan^{-1}x)$



Let; $\cot\lambda = 1 + x$

$\tan\beta = x$

$\Rightarrow \sin\lambda = \cos\beta$

$\Rightarrow \frac{1}{\sqrt{x^2+2x+2}} = \frac{1}{1\sqrt{1+x^2}}$
$\Rightarrow x^2 + 2x + 2 = x^2 + 1$
$\Rightarrow x = -1/2$

Question53

The value of $\tan^{-1}\left[\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right]$, $|x| < \frac{1}{2}$, $x \neq 0$, is equal to [Online April 8, 2017]

Options:

A. $\frac{\pi}{4} + \frac{1}{2}\cos^{-1}x^2$

B. $\frac{\pi}{4} + \cos^{-1}x^2$

C. $\frac{\pi}{4} - \frac{1}{2}\cos^{-1}x^2$

D. $\frac{\pi}{4} - \cos^{-1}x^2$

Answer: A

Solution:

Solution:

Let $x^2 = \cos 2\theta$; $\Rightarrow \theta = \frac{1}{2}\cos^{-1}x^2$

$\Rightarrow \tan^{-1}\left[\frac{\sqrt{1+x^2}}{\sqrt{1+x^2}} - \frac{\sqrt{1-x^2}}{\sqrt{1-x^2}}\right] = \left[\frac{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}\right]$

$\Rightarrow \tan^{-1}\left[\frac{1+\tan\theta}{1-\tan\theta}\right] = \tan^{-1}\left[\tan\left(\frac{\pi}{4}+\theta\right)\right]$

$= \frac{\pi}{4} + \frac{1}{2}\cos^{-1}x^2$

Question54

Let $\tan^{-1}y = \tan^{-1}x + \tan^{-1}\left(\frac{2x}{1-x^2}\right)$, where or $|x| < \frac{1}{\sqrt{3}}$. Then a value of y is : [2015]

Options:

A. $\frac{3x-x^3}{1+3x^2}$

B. $\frac{3x+x^3}{1+3x^2}$

C. $\frac{3x - x^3}{1 - 3x^2}$

D. $\frac{3x + x^3}{1 - 3x^2}$

Answer: C

Solution:

Solution:

Given that, $\tan^{-1}y = \tan^{-1}x + \tan^{-1}\left[\frac{2x}{1 - x^2}\right]$

$= \tan^{-1}x + 2\tan^{-1}x = 3\tan^{-1}x$

$\tan^{-1}y = \tan^{-1}\left[\frac{3x - x^3}{1 - 3x^2}\right]$

$\Rightarrow y = \frac{3x - x^3}{1 - 3x^2}$

Question55

If $f(x) = 2\tan^{-1}x + \sin^{-1}\left(\frac{2x}{1 + x^2}\right)$, $x > 1$ then $f(5)$ is equal to :

[Online April 10, 2015]

Options:

A. $\tan^{-1}\left(\frac{65}{156}\right)$

B. $\frac{\pi}{2}$

C. π

D. $4\tan^{-1}(5)$

Answer: C

Solution:

Solution:

$f(x) = 2\tan^{-1}x + \sin^{-1}\left(\frac{2x}{1 + x^2}\right)$

$\Rightarrow f(x) = 2\tan^{-1}x + \pi - 2\tan^{-1}x$

$\Rightarrow f(x) = \pi$

$\Rightarrow f(5) = \pi$

Question56

The principal value of $\tan^{-1}\left(\cot\frac{43\pi}{4}\right)$ is:

[Online April 19, 2014]

Options:

A. $-\frac{3\pi}{4}$

B. $\frac{3\pi}{4}$

C. $-\frac{\pi}{4}$

D. $\frac{\pi}{4}$

Answer: C

Solution:

Solution:
Consider

$\tan^{-1}\left[\cot\frac{43\pi}{4}\right] = \tan^{-1}\left[\cot\left(10\pi + \frac{3\pi}{4}\right)\right]$

$= \tan^{-1}\left[\cot\frac{3\pi}{4}\right] [\because \cot(2n\pi + \theta) = \cot\theta]$

$= \tan^{-1}\left[\tan\left(\frac{\pi}{2} - \frac{3\pi}{4}\right)\right]$

$= \frac{\pi}{2} - \frac{3\pi}{4} = \frac{2\pi - 3\pi}{4} = \frac{-\pi}{4}$

Question57

Statement I: The equation $(\sin^{-1}x)^3 + (\cos^{-1}x)^3 - a\pi^3 = 0$ has a solution for all $a \geq \frac{1}{32}$.
Statement II: For any $x \in R$, $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$ and

$0 \leq \left(\sin^{-1}x - \frac{\pi}{4}\right)^2 \leq \frac{9\pi^2}{16}$

[Online April 12, 2014]

Options:

A. Both statements I and II are true.

B. Both statements I and II are false.

C. Statement I is true and statement II is false.

D. Statement I is false and statement II is true

Answer: A

Solution:

Solution:

$\sin^{-1}x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\Rightarrow -\frac{3\pi}{4} \leq \left(\sin^{-1}x - \frac{\pi}{4}\right) \leq \frac{\pi}{4}$

$0 \le \left(\sin^{-1}x - \frac{\pi}{4}\right)^2 \le \frac{9}{16}\pi^2$

Statement II is true

$(\sin^{-1}x)^3 + (\cos^{-1}x)^3 = a\pi^3$

$\Rightarrow (\sin^{-1}x + \cos^{-1}x)[(\sin^{-1}x + \cos^{-1}x)^2 - 3\sin^{-1}x\cos^{-1}x] = a\pi^3$

$\Rightarrow \frac{\pi^2}{4} - 3\sin^{-1}x\cos^{-1}x = 2a\pi^2$

$\Rightarrow \sin^{-1}x\left(\frac{\pi}{2} - \sin^{-1}x\right) = \frac{\pi^2}{12}(1 - 8a)$

$\Rightarrow \left(\sin^{-1}x - \frac{\pi}{4}\right)^2 = \frac{\pi^2}{12}(8a - 1) + \frac{\pi^2}{16}$

$\Rightarrow \left(\sin^{-1}x - \frac{\pi}{4}\right)^2 = \frac{\pi^2}{48}(32a - 1)$

Putting this value in equation

$0 \le \frac{\pi^2}{48}(32a - 1) \le \frac{9}{16}\pi^2$

$\Rightarrow 0 \le 32a - 1 \le 27$

$\frac{1}{32} \le a \le \frac{7}{8}$

Statement-I is also true

Question58

The number of solutions of the equation, $\sin^{-1}x = 2\tan^{-1}x$ (in principal values) is:
[Online April 22, 2013]

Options:

A. 1

B. 4

C. 2

D. 3

Answer: A

Solution:

Solution:

Given equation is

$\sin^{-1}x = 2\tan^{-1}x$

Now, this equation has only one solution.

$LHS = \sin^{-1}1 = \frac{\pi}{2}$

and RHS $= 2\tan^{-1}1 = 2 \times \frac{\pi}{4} = \frac{\pi}{2}$

Also, $x = 1$ gives angle value as $\frac{\pi}{4}$ and $\frac{5\pi}{4}$

$\frac{5\pi}{4}$ is outside the principal value.

Question59

If x, y, z are in A.P. and $\tan^{-1}x$, $\tan^{-1}y$ and $\tan^{-1}z$ are also in A.P., then

[2013]

Options:

A. $x = y = z$

B. $2x = 3y = 6z$

C. $6x = 3y = 2z$

D. $6x = 4y = 3z$

Answer: A

Solution:

Solution:
Since, x, y, z are in A.P.
$\therefore 2y = x + z$
Also, we have
$2\tan^{-1}y = \tan^{-1}x + \tan^{-1}(z)$
$\Rightarrow \tan^{-1}\left(\frac{2y}{1-y^2}\right) = \tan^{-1}\left(\frac{x+z}{1-xz}\right)$
$\Rightarrow \frac{x+z}{1-y^2} = \frac{x+z}{1-xz} (\because 2y = x + z)$
$\Rightarrow y^2 = xz$ or $x + z = 0 \Rightarrow x = y = z = 0$

Question60

Let $x \in (0, 1)$. The set of all x such that $\sin^{-1}x > \cos^{-1}x$, is the interval: [Online April 25, 2013]

Options:

A. $\left(\frac{1}{2}, \frac{1}{\sqrt{2}}\right)$

B. $\left(\frac{1}{\sqrt{2}}, 1\right)$

C. (0,1)

D. $\left(0, \frac{\sqrt{3}}{2}\right)$

Answer: B

Solution:

Solution:
Given $\sin^{-1}x > \cos^{-1}x$ where $x \in (0, 1)$
$\Rightarrow \sin^{-1}x > \frac{\pi}{2} - \sin^{-1}x$
$\Rightarrow 2\sin^{-1}x > \frac{\pi}{2} \Rightarrow \sin^{-1}x > \frac{\pi}{4}$
$\Rightarrow x > \sin\frac{\pi}{4} \Rightarrow x > \frac{1}{\sqrt{2}}$

Maximum value of $\sin^{-1}x$ is $\frac{\pi}{2}$

So, maximum value of x is 1. So, $x \in \left(\frac{1}{\sqrt{2}}, 1\right)$.

Question61

$S = \tan^{-1}\left(\frac{1}{n^2+n+1}\right) + \tan^{-1}\left(\frac{1}{n^2+3n+3}\right) + \ldots.. + \tan^{-1}\left(\frac{1}{1+(n+19)(n+20)}\right)$, then $\tan S$ is equal to:
[Online April 23, 2013]

Options:

A. $\frac{20}{401+20n}$

B. $\frac{n}{n^2+20n+1}$

C. $\frac{20}{n^2+20n+1}$

D. $\frac{n}{401+20n}$

Answer: C

Solution:

Solution:
We know that,

$\tan^{-1}\frac{1}{1+2} + \tan^{-1}\frac{1}{1+2\times 3} + \tan^{-1}\frac{1}{1+3\times 4} + \ldots\ldots. + \tan^{-1}\frac{1}{1+(n-1)n} + \tan^{-1}\frac{1}{1+n(n+1)} + \ldots\ldots.. +$

$\tan^{-1}\frac{1}{1+(n+19)(n+20)} = \tan^{-1}\frac{n+19}{n+21}$

$\Rightarrow \tan^{-1}\frac{n-1}{n+1} + \tan^{-1}\frac{1}{1+n(n+1)} + \tan^{-1}\frac{1}{1+(n+1)(n+2)} + \ldots.. + \frac{1}{1+(n+19)(n+20)} = \tan^{-1}\frac{n+19}{n+21}$

$\Rightarrow \tan^{-1}\frac{1}{1+n(n+1)} + \tan^{-1}\frac{1}{1+(n+1)(n+2)} + \ldots\ldots. + \frac{1}{1+(n+19)(n+20)} = \tan^{-1}\frac{n+19}{n+21} - \tan^{-1}\frac{n-1}{n+1}$

$\Rightarrow \tan^{-1}\left(\frac{1}{n^2+n+1}\right) + \tan^{-1}\left(\frac{1}{n^2+3n+3}\right) + \ldots\ldots. + \tan^{-1}\frac{1}{1+(n+19)(n+20)}$

$= \tan^{-1}\left(\frac{\frac{n+19}{n+21} - \frac{n-1}{n+1}}{1 + \frac{n+19}{n+21} \times \frac{n-1}{n+1}}\right) = \tan^{-1}\frac{20}{n^2+20n+1} = S$

$\therefore \tan^{-1}S = \frac{20}{n^2+20n+1}$

Question62

A value of x for which $\sin(\cot^{-1}(1+x)) = \cos(\tan^{-1}x)$, is
[Online April 9, 2013]

Options:

A. $-\frac{1}{2}$

B. 1

C. 0

D. $\frac{1}{2}$

Answer: A

Solution:

Solution:
$\sin(\cot^{-1}(1+x)) = \cos(\tan^{-1}x)$
$\Rightarrow \operatorname{cosec}^2(\cot^{-1}(1+x)) = \sec^2(\tan^{-1}x)$
$\Rightarrow 1 + [\cot(\cot^{-1}(1+x))]^2 = 1 + [\tan(\tan^{-1}x)]^2$
$\Rightarrow (1+x)^2 = x^2 \Rightarrow x = -\frac{1}{2}$

Question63

A value of $\tan^{-1}\left(\sin\left(\cos^{-1}\left(\sqrt{\frac{2}{3}}\right)\right)\right)$ is

[Online May 19, 2012]

Options:

A. $\frac{\pi}{4}$

B. $\frac{\pi}{2}$

C. $\frac{\pi}{3}$

D. $\frac{\pi}{6}$

Answer: D

Solution:

Solution:
Consider $\tan^{-1}\left[\sin\left(\cos^{-1}\sqrt{\frac{2}{3}}\right)\right]$
Let $\cos^{-1}\sqrt{\frac{2}{3}} = \theta \Rightarrow \cos\theta = \sqrt{\frac{2}{3}}$
$\Rightarrow \sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\frac{2}{3}} = \sqrt{\frac{1}{3}}$
$\therefore \tan^{-1}\left[\sin\left(\cos^{-1}\sqrt{\frac{2}{3}}\right)\right] = \tan^{-1}[\sin\theta]$
$= \tan^{-1}\left[\sqrt{\frac{1}{3}}\right] = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

Question64

The largest interval lying in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ for which the function,

$f(x) = 4^{-x^2} + \cos^{-1}\left(\frac{x}{2} - 1\right) + \log(\cos x)$, is defined, is

[2007]

Options:

A. $\left[-\frac{\pi}{4}, \frac{\pi}{2}\right)$

B. $\left[0, \frac{\pi}{2}\right)$

C. $[0, \pi]$

D. $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Answer: B

Solution:

Solution:
Given that
$f(x) = 4^{-x^2} + \cos^{-1}\left(\frac{x}{2} - 1\right) + \log(\cos x)$

$f(x)$ is defined if $-1 \le \left(\frac{x}{2} - 1\right) \le 1$ and $\cos x > 0$

$\Rightarrow 0 \le \frac{x}{2} \le 2$ and $-\frac{\pi}{2} < x < \frac{\pi}{2}$

$\Rightarrow 0 \le x \le 4$ and $-\frac{\pi}{2} < x < \frac{\pi}{2}$

$\therefore x \in \left[0, \frac{\pi}{2}\right)$

Question65

If $\sin^{-1}\left(\frac{x}{5}\right) + \text{cosec}^{-1}\left(\frac{5}{4}\right) = \frac{\pi}{2}$, then the values of x is

[2007]

Options:

A. 4

B. 5

C. 1

D. 3

Answer: D

Solution:

Solution:

$\sin^{-1}\left(\frac{x}{5}\right) + \text{cosec}^{-1}\left(\frac{5}{4}\right) = \frac{\pi}{2}$

$\Rightarrow \sin^{-1}\left(\frac{x}{5}\right) = \frac{\pi}{2} - \text{cosec}^{-1}\left(\frac{5}{4}\right)$

$\Rightarrow \sin^{-1}\left(\frac{x}{5}\right) = \frac{\pi}{2} - \sin^{-1}\left(\frac{4}{5}\right)$ $[\because \sin^{-1}x + \cos^{-1}x = \pi / 2]$

$\Rightarrow \sin^{-1}\left(\frac{x}{5}\right) = \cos^{-1}\left(\frac{4}{5}\right)$

$\sin^{-1}\frac{x}{5} = \sin^{-1}\sqrt{1 - \left(\frac{4}{5}\right)^2}$ $[\because \cos^{-1}x = \sin^{-1}\sqrt{1 - x^2}]$

$\Rightarrow \sin^{-1}\frac{x}{5} = \sin^{-1}\frac{3}{5} \Rightarrow \frac{x}{5} = \frac{3}{5}$

$\Rightarrow x = 3$

Question66

If $\cos^{-1}x - \cos^{-1}\frac{y}{2} = \alpha$, then $4x^2 - 4xy\cos\alpha + y^2$ is equal to
[2005]

Options:

A. $2\sin 2\alpha$

B. 4

C. $4\sin^2\alpha$

D. $-4\sin^2\alpha$

Answer: C

Solution:

Solution:

$\cos^{-1}x - \cos^{-1}\frac{y}{2} = \alpha$

$\Rightarrow \cos^{-1}\left(\frac{xy}{2} + \sqrt{(1 - x^2)\left(1 - \frac{y^2}{4}\right)}\right) = \alpha$

$\Rightarrow \cos^{-1}\left(\frac{xy + \sqrt{4 - y^2 - 4x^2 + x^2y^2}}{2}\right) = \alpha$

$\Rightarrow xy + \sqrt{4 - y^2 - 4x^2 + x^2y^2} = 2\cos\alpha$

$\Rightarrow \sqrt{4 - y^2 - 4x^2 + x^2y^2} = 2\cos\alpha - xy$

Squaring both sides, we get

$\Rightarrow 4 - y^2 - 4x^2 + x^2y^2 = 4\cos^2\alpha + x^2y^2 - 4xy\cos\alpha$

$\Rightarrow 4x^2 + y^2 - 4xy\cos\alpha = 4\sin^2\alpha$

Question67

The domain of the function $f(x) = \frac{\sin^{-1}(x - 3)}{\sqrt{9 - x^2}}$
[2004]

Options:

A. [1, 2]

B. [2, 3)

C. [1, 2]

D. [2, 3]

Answer: B

Solution:

Solution:

$f(x) = \frac{\sin^{-1}(x-3)}{\sqrt{9-x^2}}$ is defined

When $-1 \le x - 3 \le 1 \Rightarrow 2 \le x \le 4$ ....(i)

and $9 - x^2 > 0 \Rightarrow -3 < x < 3$ ......(ii)

from (i) and (ii),

we get $2 \le x < 3 \therefore$ Domain $= [2, 3)$

Question68

The trigonometric equation $\sin^{-1}x = 2\sin^{-1}a$ has a solution for [2003]

Options:

A. $|a| \le \frac{1}{\sqrt{2}}$

B. $\frac{1}{2} < |a| < \frac{1}{\sqrt{2}}$

C. all real values of a

D. $|a| < \frac{1}{2}$

Answer: A

Solution:

Solution:

Given that $\sin^{-1}x = 2\sin^{-1}a$

We know that $-\frac{\pi}{2} \le \sin^{-1}x \le \frac{\pi}{2}$

$\Rightarrow -\frac{\pi}{2} \le 2\sin^{-1}a \le \frac{\pi}{2}$

$\Rightarrow -\frac{\pi}{4} \le \sin^{-1}a \le \frac{\pi}{4} \Rightarrow \frac{-1}{\sqrt{2}} \le a \le \frac{1}{\sqrt{2}}$

$\therefore |a| \le \frac{1}{\sqrt{2}}$

Question69

$\cot^{-1}(\sqrt{\cos\alpha}) - \tan^{-1}(\sqrt{\cos\alpha}) = x$, then $\sin x =$
[2002]

Options:

A. $\tan^2\left(\frac{\alpha}{2}\right)$

B. $\cot^2\left(\frac{\alpha}{2}\right)$

C. $\tan\alpha$

D. $\cot\left(\frac{\alpha}{2}\right)$

Answer: A

Solution:

Solution:
Given that, $\cot^{-1}(\sqrt{\cos\alpha}) - \tan^{-1}(\sqrt{\cos\alpha}) = x$

$\tan^{-1}\left(\frac{1}{\sqrt{\cos\alpha}}\right) - \tan^{-1}(\sqrt{\cos\alpha}) = x$

$\Rightarrow \tan^{-1}\frac{\frac{1}{\sqrt{\cos\alpha}} - \sqrt{\cos\alpha}}{1 + \frac{1}{\sqrt{\cos\alpha}}.\sqrt{\cos\alpha}} = x$

$\Rightarrow \tan^{-1}\frac{1-\cos\alpha}{2\sqrt{\cos\alpha}} = x$

$\Rightarrow \tan x = \frac{1-\cos\alpha}{2\sqrt{\cos\alpha}}$

$\Rightarrow \cot x = \frac{2\sqrt{\cos\alpha}}{1-\cos\alpha} = \frac{B}{P}$

$P = (1-\cos\alpha)$ and $B = 2\sqrt{\cos\alpha}$

$H = \sqrt{P^2 + B^2} = 1 + \cos\alpha$

$\Rightarrow \sin x = 1 - \cos\alpha 1 + \cos\alpha = \frac{1-(1-2\sin^2\alpha/2)}{1+2\cos^2\alpha/2-1}$

or $\sin x = \tan^2\frac{\alpha}{2}$

Question70

The domain of $\sin^{-1}[\log_3(x/3)]$ is
[2002]

Options:

A. [1,9]

B. [-1,9]

C. [-9,1]

D. [-9,-1]

Answer: A

Solution:

$f(x) = \sin^{-1}\left(\log_3\left(\frac{x}{3}\right)\right)$

We know that domain of $\sin^{-1}x$ is $x \in [-1, 1]$

$\therefore -1 \le \log_3\left(\frac{x}{3}\right) \le 1 \Rightarrow 3^{-1} \le \frac{x}{3} \le 3^1$

$\Rightarrow 1 \le x \le 9$ or $x \in [1, 9]$
